Nonlinear Optimization of the Shape Functions in the Finite Element Method When Determining Cutoff Frequencies of Waveguides of Arbitrary Cross Section

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Abstract —**The present paper deals with a review of the recently developed k optimization process of the finite element method when solving eigenvalue problems. The methodology is then applied to the determination of the fundamental cutoff frequency of a hollow-piped waveguide of cardioidal cross section. It is shown that a considerable reduction in computer memory and/or CPU time is achieved.**

I. Introduction

As stated recently by Kuttler [1] in an excellent paper, "many important waveguides have complicated cross sections which cannot be solved by the method of separation of variables. A variety of approximation methods have been used to try to determine the cutoff frequencies of such waveguides."

Among the many approximate analytical methods, the methods of Galerkin, Rayleigh, and Ritz are perhaps the best known. On the other hand, they do constitute the essential foundation to one of the most popular and universally used computational algorithms: the finite element method.

Minimizing the discretization and numerical errors is certainly a question of the utmost importance when using the finite element (FE) method in order to ascertain reliable results. On the other hand, from economic and scientific viewpoints, the analyst wishes to accomplish these goals without increasing computer memory and/or CPU time.

Rather recently, the concept of the k optimization parameter contained in the shape functions [2] has been developed.[1] The procedure consists of including an unknown, exponential parameter k in the shape functions when determining natural frequencies or critical loads when the FE method is formulated by using the Rayleigh–Ritz approach. Since this formulation yields upper bounds [4], by numerical minimization of the eigenvalues with respect to the parameter k, one is able to optimize the eigenvalues under study.

This paper presents a brief discussion of the method and its application to a waveguide of cardioidal cross section (Fig. 1). Eigenvalues for this type of complicated cross section have been determined by several methods in [5] and this fact allows for a reasonable assurance of the accuracy of the numerical results presented here. On the other hand the cardioidal shape resembles rather closely the "heart-shaped" waveguide studied extensively by Daly [6].

II. Finite Element Formulation

The elements are quadrilaterals with straight sides, four nodes, and one degree of freedom per node, as shown in Fig. 2. For each

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[1] Successful efforts in this direction were simultaneously achieved by Bert and coworkers [3].

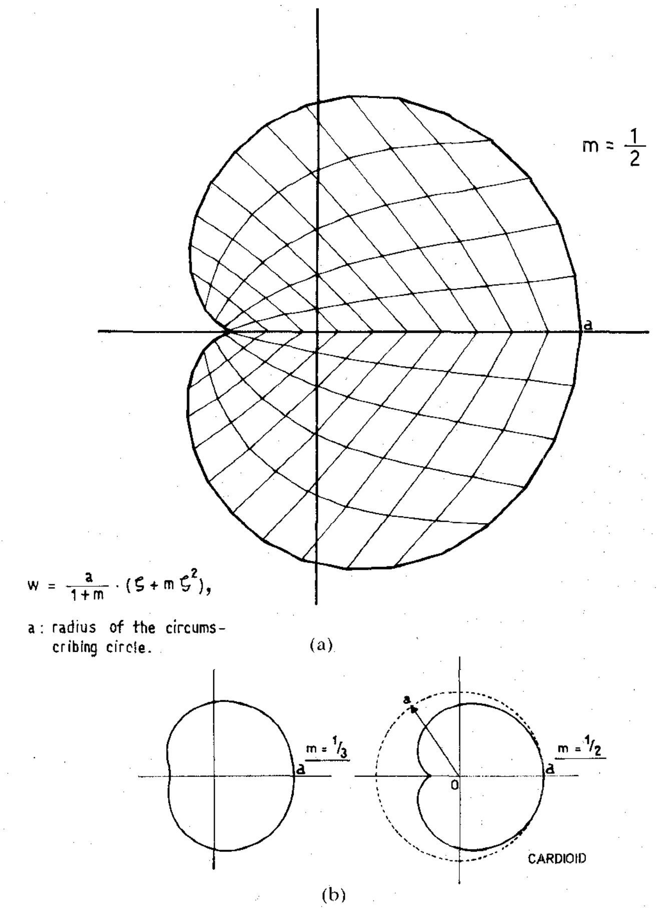

Fig. 1. Cardioidal domain. (a) Finite element mesh for the cardioid ($m = 1/2$; 100 quadrilateral elements, 121 nodes, and 81 degrees of freedom). (b) Configurations mapped into a unit circle by eq. (4).

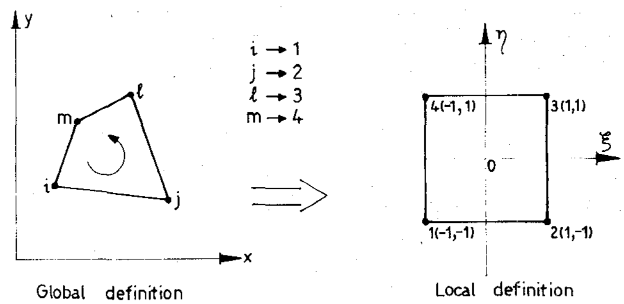

Fig. 2. Description of the quadrilateral elements.

node, the following shape functions, which depend on a nonlinear parameter k to be optimized, are defined:

$$N_1(\xi,\eta;\ k) = 1 - \left[1 - \tfrac{1}{4}(1-\xi)(1-\eta)\right]^k$$

$$N_2(\xi,\eta;\ k) = 1 - \left[1 - \tfrac{1}{4}(1+\xi)(1-\eta)\right]^k$$

$$N_3(\xi,\eta;\ k) = 1 - \left[1 - \tfrac{1}{4}(1+\xi)(1+\eta)\right]^k$$

$$N_4(\xi,\eta;\ k) = 1 - \left[1 - \tfrac{1}{4}(1-\xi)(1+\eta)\right]^k. \qquad (1)$$

The procedure consists basically in assuming an approximation of each eigenfunction $U(x,y)$ of the 2-D Helmholtz equation by a linear combination $U^h(x,y;\ k)$ of the shape functions that

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corresponds to every node:

$$U(x,y) \cong U^h(x,y;\,k) = \sum_{i=1}^{n} N_i(x,y;\,k)\,u_i = \underline{N}(x,y;\,k)\,\underline{u} \tag{2}$$

where n is the total number of nodes of the mesh. The u_i's are the nodal values of U^h.

Taking a fixed value of the k parameter, the standard Rayleigh–Ritz procedure leads to a generalized algebraic eigenvalue problem of the form

$$\boldsymbol{S}(k)\,\underline{u} = \omega^2 \boldsymbol{M}(k)\,\underline{u} \tag{3}$$

where ω is the frequency (dependent upon the k parameter) associated with each eigenfunction (approximated by the eigenvector u). $\boldsymbol{S}$ (the "stiffness" matrix) and $\boldsymbol{M}$ (the "mass" matrix) are both symmetric square matrices with band structure. Also, $\boldsymbol{M}$ is positive definite. Gauss numerical quadrature is employed in order to calculate all the integrals contained in both matrices, taking four sample points for each coordinate (16 points per element). The Jacobian matrix and its inverse are calculated in a straightforward manner.

The eigenvalue problem (3) is solved by a combination of the inverse power iteration and the Rayleigh quotient iteration methods. This combination acts directly on both matrices without any kind of factorization. This algorithm has been summarized in [2].

Since the Rayleigh–Ritz approach yields upper bounds of the eigenvalues, by numerical minimization of the eigenvalues with respect to the k parameter one is able to optimize the eigenvalues under study. This minimization requires that the explained procedure be repeated for a set of values of the k parameter, and this procedure increases extensively the computational work. Nevertheless, the numerical experiments performed in the present investigation show that a drastic reduction in memory and/or computer time can be achieved by this new procedure in the case of domains of complicated boundary shape.

III. Numerical Results

Consider hollow waveguides whose cross section can be conformally transformed onto a unit circle by the functional relation

$$w = A(\zeta + m\zeta^2) \tag{4}$$

where A is a scale factor and m is a parameter such that $m \leqslant \frac{1}{2}$. For $m = \frac{1}{2}$ one has the cardioid shape (see Fig. 1).

Extensive numerical results are available in [5] for the case of TM modes using a) the Galerkin method coupled with conformal transformation of the given domain onto a unit circle and b) a classical finite element formulation which makes use of triangular elements and one unknown per node (A). These results are shown in Table I and compared with the results obtained in the present investigation. In column (B) quadrilateral elements are used with linear base functions, and in (C) the k optimization process is carried out.

It is observed that one can achieve almost as much accuracy as that attained in [5], where 576 triangular elements and 276 unknowns were used, employing only 100 quadrilateral elements and 81 unknown when the k optimization procedure is used.

TABLE I
Comparison of Fundamental Eigenvalues Determined by Several Approaches

	Ω_{01}				
m	Analytical Results [5]	(A)	(B)	(C)	k_{opt}
0.001	2.407	2.411	2.4252	2.4159	1.0356
0.01	2.428	2.433	2.4468	2.4374	1.0355
0.025	2.463	2.467	2.4819	2.4723	1.0355
0.1	2.621	2 625	2.6399	2.6296	1.0356
0.2	2.787	2.787	2.8045	2.7929	1 0359
0.25	2 851	2.849	2 8685	2.8563	1.0361
0 5	3 039	3.036	3.0640	3.0480	1.0361

(A): Finite element results; 576 triangular elements and 276 unknowns [5].

(B): Finite element results; 100 quadrilateral elements and 81 unknowns ($k = 1$).

(C): Same as (B) but the eigenvalues are determined using the k optimization process.

Similar calculations of the fundamental cutoff frequency of a waveguide of square cross section (which has an exact solution) have been performed in the present study in order to investigate the economies attained in memory requirements and CPU time. These calculations have demonstrated that, for example, using 225 degrees of freedom with bilinear shape functions ($k = 1$) one obtains the first eigenvalue with a relative accuracy of 0.32 percent with a CPU time consumed of 500 arbitrary units. Optimizing now with respect to the k parameter with four global iterations using a mesh with only 28 degrees of freedom, one can achieve the same relative accuracy consuming 280 arbitrary units of CPU time.

Then, the efficiency ratio of the present approach is, for that problem, 1.80 in CPU time and a value of 8 from the point of view of the number of degrees of freedom.

In conclusion, it appears that the present optimization process possesses attractive features from scientific and economic viewpoints. In principle it may also be applied to eigenvalue problems in anisotropic media, nonlinear situations, higher order nodes, etc.

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